ON THE OSCILLATION OF THE BROWNIAN MOTION PROCESS*

BY

ARYEH DVORETZKY

ABSTRACT

Paley, Wiener and Zygmund proved that, with probability 1, Brownian paths never satisfy a Lipschitz condition of order greater than 1/2. This result is improved by showing that they never satisfy even a Lipschitz condition of order 1/2 with a sufficiently small Lipschitz constant.

1. **Introduction.** Let $x(t) = x_\omega(t)$, $-\infty < t < \infty$, be the sample functions of a separable Brownian motion process, i.e. a stochastic process with $x(0) \equiv 0$, having independent increments $x(t) - x(s)$ normally distributed with mean 0 and variance $|t-s|$, and with almost surely continuous sample functions. Already Paley, Wiener and Zygmund [1] proved that with probability 1 the sample functions satisfy nowhere a one-sided Lipschitz condition of order greater than 1/2; more precisely they established

$$(1) \qquad P\left\{ \inf_{-\infty<t<\infty} \limsup_{h\to 0+} \frac{|x(t+h)-x(t)|}{h^{\frac{1}{2}+\varepsilon}} = \infty \right\} = 1$$

for every $\varepsilon > 0$.

J.-P. Kahane drew our attention to this result and asked whether the ε in (1) can be dropped. As written we cannot prove (1) with $\varepsilon = 0$, but if we write the above result in the equivalent form obtained by replacing $= \infty$ in (1) with > 0, then it remains valid even for $\varepsilon = 0$. We shall indeed prove something more, namely the following

THEOREM. *There exists a universal* $c > 0$ *such that*

$$(2) \qquad P\left\{ \inf_{-\infty<t<\infty} \limsup_{h\to 0+} \frac{|x(t+h)-x(t)|}{h^{\frac{1}{2}}} < c \right\} = 0.$$

2. **Proof.** It is obviously enough to establish (2) with the inf taken over $0 \leqq t \leqq 1$ instead of over all real t. This assertion is in turn implied by

$$(3) \quad P\{|x(t+h)-x(t)| < ch^{\frac{1}{2}} \text{ for all } 0 \leqq h \leqq \Delta \text{ and at least one } t \text{ in } [0,1]\} = 0$$

for all $\Delta > 0$. Since it is enough to prove this for a sequence of values of Δ tend-

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ing to 0, the assertion would follow once we exhibit a $c > 0$ having the property that (3) holds for any given $\Delta > 0$.

Let n be a positive integer and denote by $A_i^{(n)}(\Delta)$ $(i = 1,2,\dots,n)$ the event

$$(4) \qquad |x(t+h) - x(t)| < ch^{1/2} \text{ for all } 0 \leqq h \leqq \Delta \text{ and at least one } t \text{ in } \left[\frac{i-1}{n}, \frac{i}{n}\right].$$

To prove the theorem we shall show that

$$(5) \qquad \lim_{n=\infty} P\left\{\bigcup_{i=1}^{n} A_i^{(n)}(\Delta)\right\} = 0.$$

Since the probability of $A_i^{(n)}(\Delta)$ does not depend on i, (5) is implied by

$$(6) \qquad \lim_{n=\infty} nP\{A_1^{(n)}(\Delta)\} = 0.$$

But if $A_1^{(n)}(\Delta)$ occurs and t is a value in $[0,1/n]$ for which (4) holds and if we denote by y the corresponding $x(t)$ then we have

$$\left|x\left(\frac{2^j}{n}\right) - y\right| \leqq c\left(\frac{2^j}{n} - t\right)^{1/2} \leqq c\left(\frac{2^j}{n}\right)^{1/2} \text{ provided } \frac{2^j}{n} - t \leqq \Delta.$$

Thus the occurrence of $A_1^{(n)}(\Delta)$ entails

$$(7) \qquad \left|x\left(\frac{2^j}{n}\right) - x\left(\frac{2^{j-1}}{n}\right)\right| < 2c\left(\frac{2^j}{n}\right)^{1/2} \text{ for } j = 1,2,\dots,\left[\frac{\log n\Delta}{\log 2}\right].$$

Since the increments are independent the probability of (7) equals the product of the probabilities

$$P\left\{\left|x\left(\frac{2^j}{n}\right) - x\left(\frac{2^{j-1}}{n}\right)\right| < 2c\left(\frac{2^j}{n}\right)^{1/2}\right\} =$$

$$= \frac{2}{(2^j\pi/n)^{1/2}} \int_0^{2c(2^j/n)^{1/2}} e^{-u^2 n/2^j}\,du = \frac{2}{(2\pi)^{1/2}} \int_0^{2^{3/2}c} e^{-u^2/2}\,du.$$

Let us now choose $c > 0$ so that

$$(8) \qquad \eta = \left(\frac{2}{\pi}\right)^{1/2} \int_0^{2^{3/2}c} e^{-u^2/2}\,du < \frac{1}{2}.$$

Then

$$nP\{A_1^{(n)}(\Delta)\} \leqq n\eta^{(\log n\Delta)/\log 2} = \eta^{\log\Delta/\log 2}\, n^{1 + (\log\eta/\log 2)}$$

and thus, by (8), tends to zero for every $\Delta > 0$. This establishes (7) and completes the proof of the theorem.

3. **Remarks** The choice of a geometric progression in (7) was made to facilitate the computation. In view of (8) this computation shows that $c = 2^{-1}(\pi/2)^{1/4}\, 2^{-3/2} = \pi^{1/4}/8$ *may be taken as the c in the theorem.* Instead of considering the sequence of points $2^j/n$ in (7) we could have considered any sequence q^j/n with $q > 1$. This would replace the condition (8) by

$$\left(\frac{2}{\pi}\right)^{1/2} \int_0^{2c(1-(1/q))^{-1/2}} e^{-u^2/2}\,du < \frac{1}{q}. \tag{9}$$

Thus any c for which there exists $q > 1$ satisfying (9) will do. A little computation shows this requirement as equivalent to

$$c < (8\pi)^{-1/4} \max_{0<t<\infty} t\left(\int_t^{\infty} e^{-u^2/2}\,du\right) = 0.28.\ldots \tag{10}$$

Therefore, *the assertion of the theorem holds for any c satisfying* (10).

For all we know the theorem may hold even with $c = \infty$. To disprove this it will be necessary to show the existence of values of t in whose vicinity the oscillations are small. But whereas the method of this note and others current in the literature are well adapted to show that oscillations cannot be too small they seem inadequate to trap "points of small oscillation". Thus, it is well known that almost surely $\liminf_{h\to 0+} |x(h)|(2h\log\log 1/h)^{-1/2} = 1$ and hence that almost surely $\limsup_{h\to 0+} |x(t+h) - x(t)|(2h\log\log 1/h)^{-1/2} = 1$ for almost all t (in the usual Lebesgue sense). It is extremely likely that the infimum of this lim sup taken over all real t is < 1 (indeed 0), i.e. that for almost all sample functions there exist values of $t = t(\omega)$ for which this lim sup is smaller than 1. But we do not know how to prove even this assertion which is so much weaker than the statement that our theorem fails with $c = \infty$.

References

1. Paley, R. E. A., Wiener, N. and Zygmund A., 1933, Notes on random functions, *Math. Z.*, 37, 647–668.

Columbia University, New York,
and
The Hebrew University of Jerusalem